Basics of Radiative Transfer / Atmosphere Modeling

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Principal Reference

Stellar Atmospheres

Mihalas (1978)

These notes are provided to help facilitate these lectures. It likely that there are typographical errors, although every endeavor has been made to minimize these.

Before using any formula you should check original source (or derive them). In some cases, rigor may have been sacrificed for clarity.

If you fine an error, please EMAIL me at hillier@pitt.edu

Introduction

Stellar Spectra:

Primarily determined by:

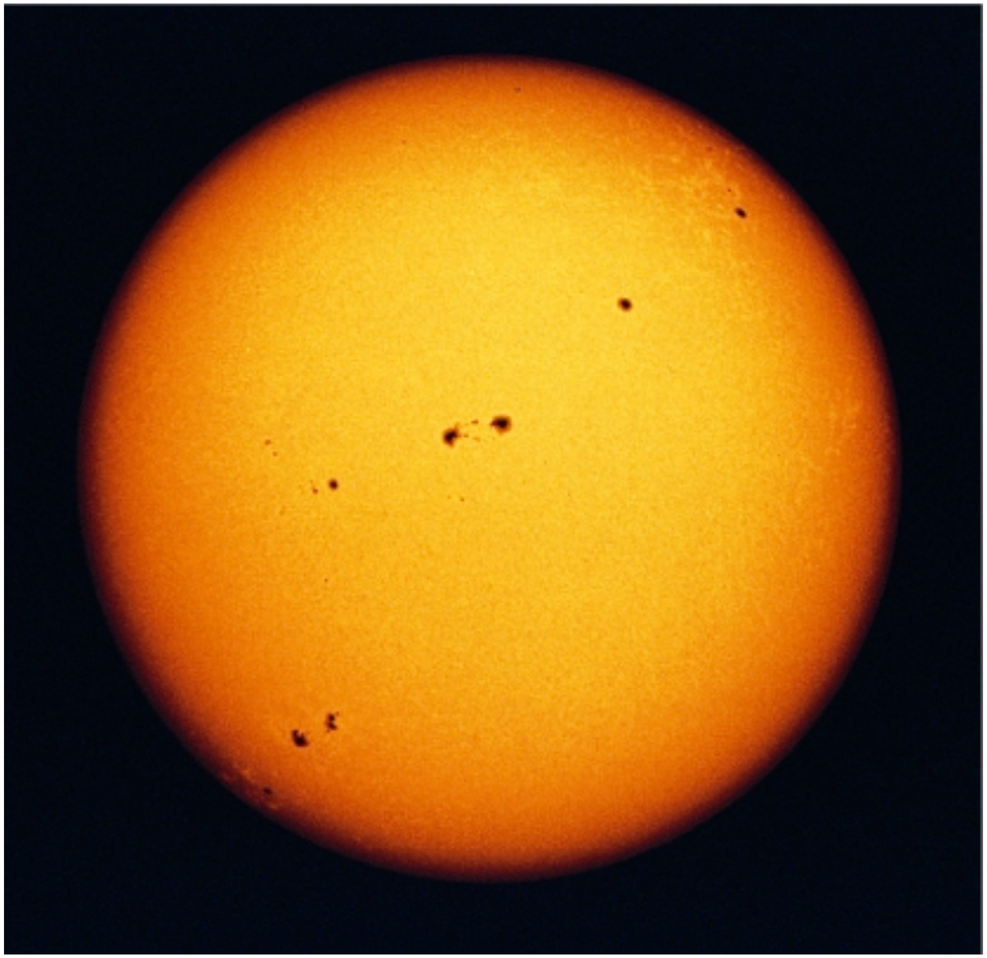
- Surface Temperature (effective temperature)
- Surface gravity
- Composition
- (Rotation)

Other (generally weaker) influences

- Magnetic activity
- Size

Question

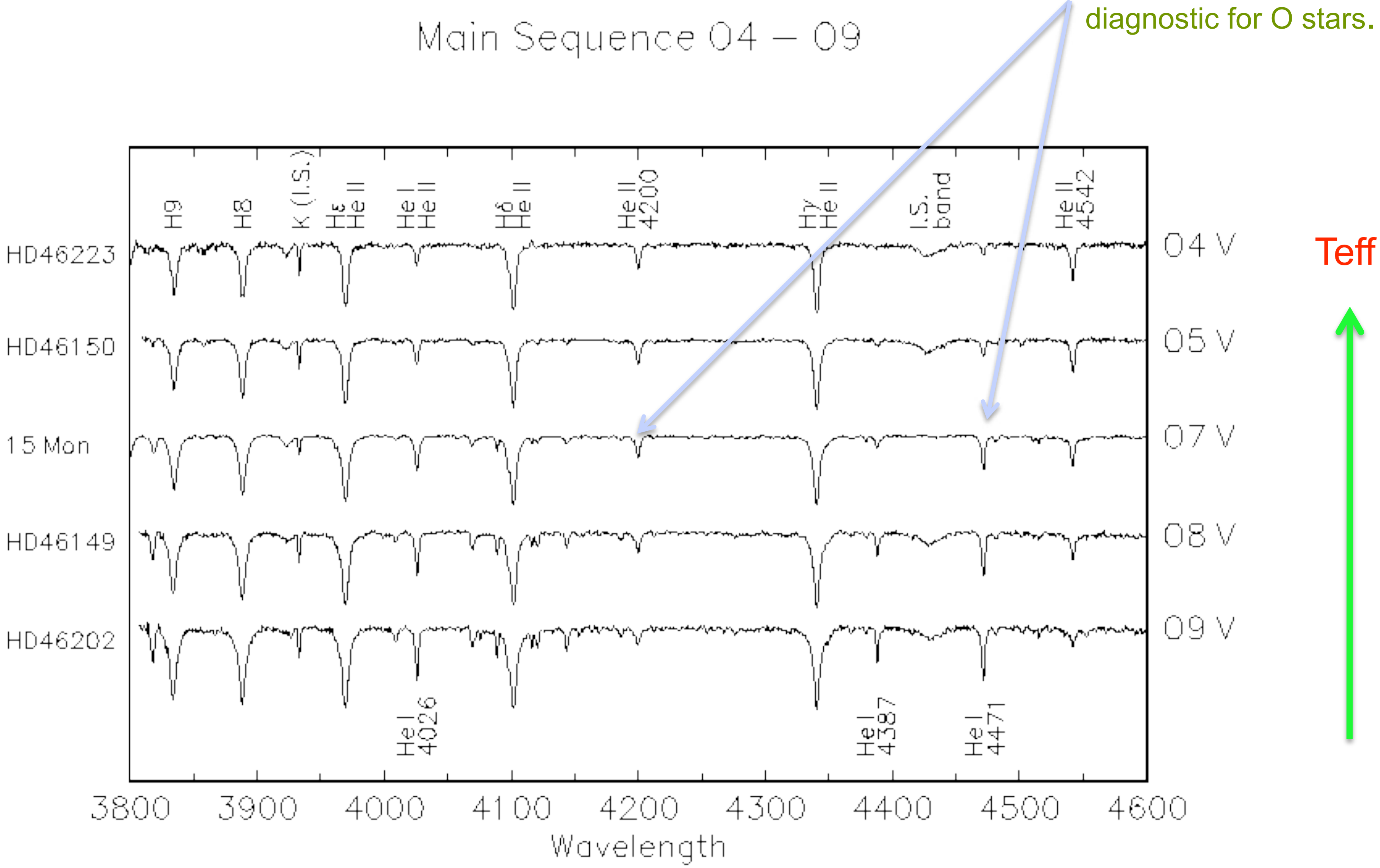
- When we look at the Sun, what do we see?
- Why?



Why does sun look like a ball?
Why is Sun darker towards the limb?
Does the Sun look like a ball at all Wavelengths? (Why/Why not)?
Is the sun darker towards limb at all wavelengths?

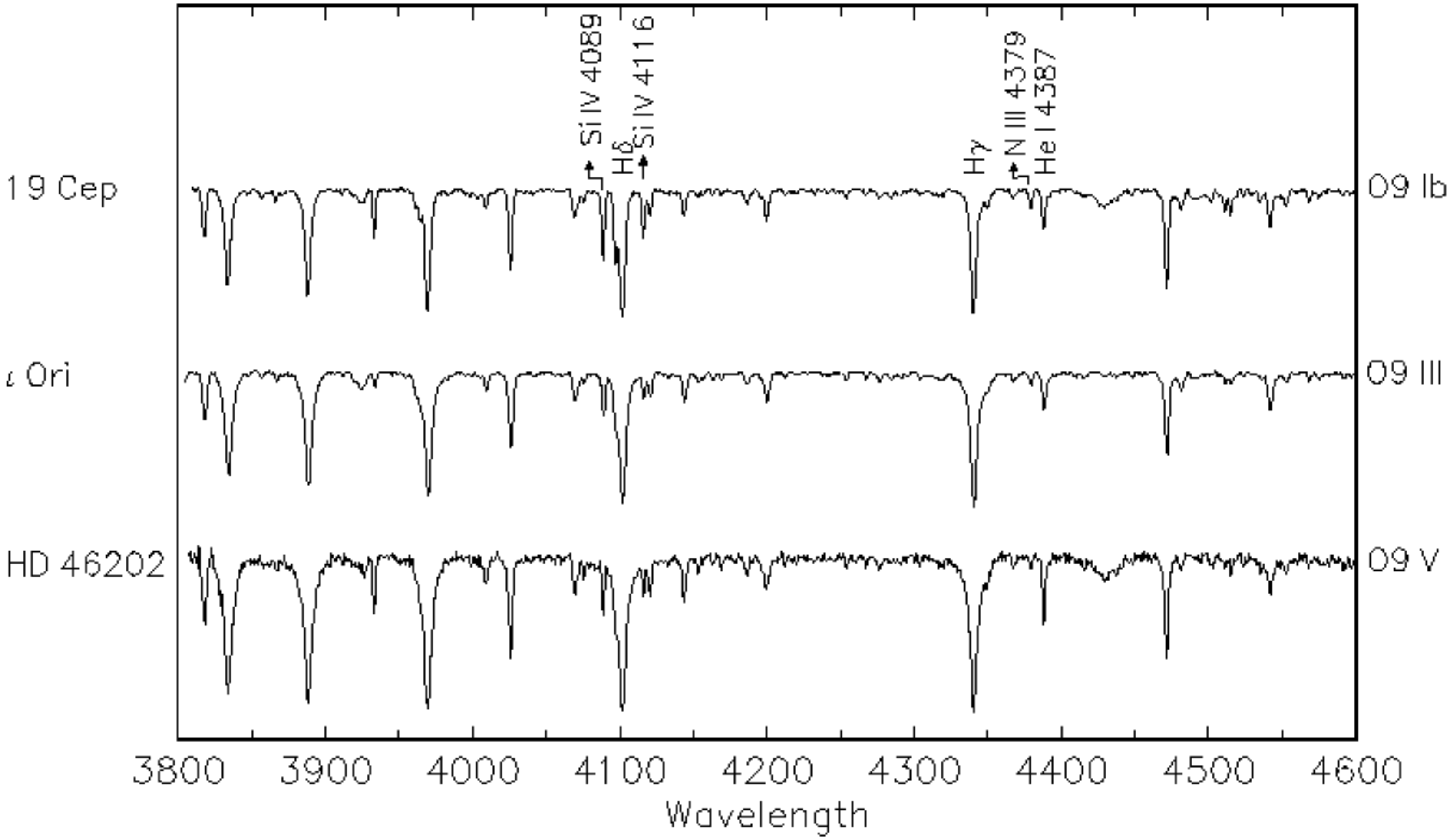
STELLAR CLASIFICATION

He I/He II: Key diagnostic for O stars.



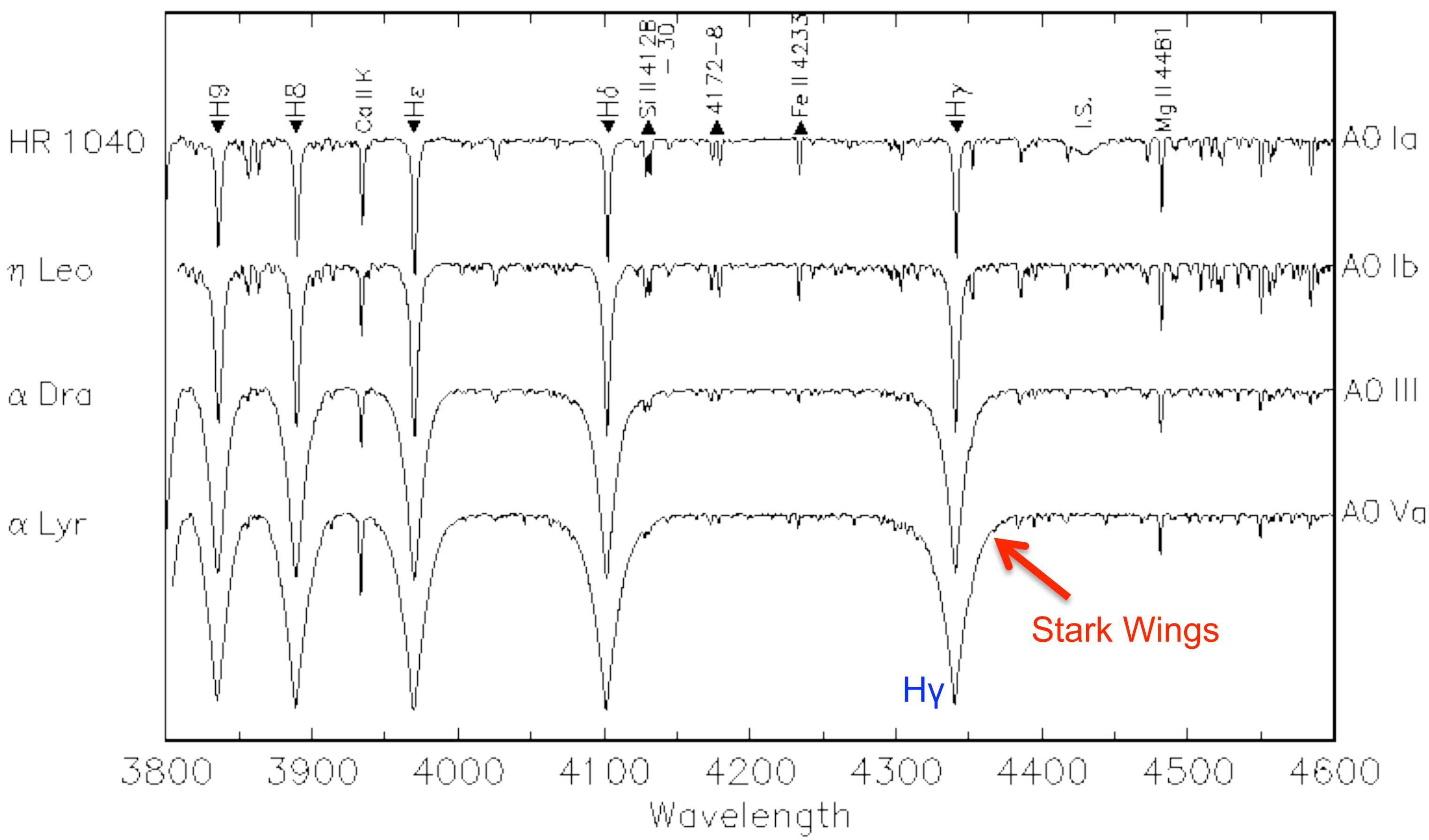
From: http://nedwww.ipac.caltech.edu/level5/Gray/frames.html

Luminosty Effects ⇔ Surface Gravity Effects



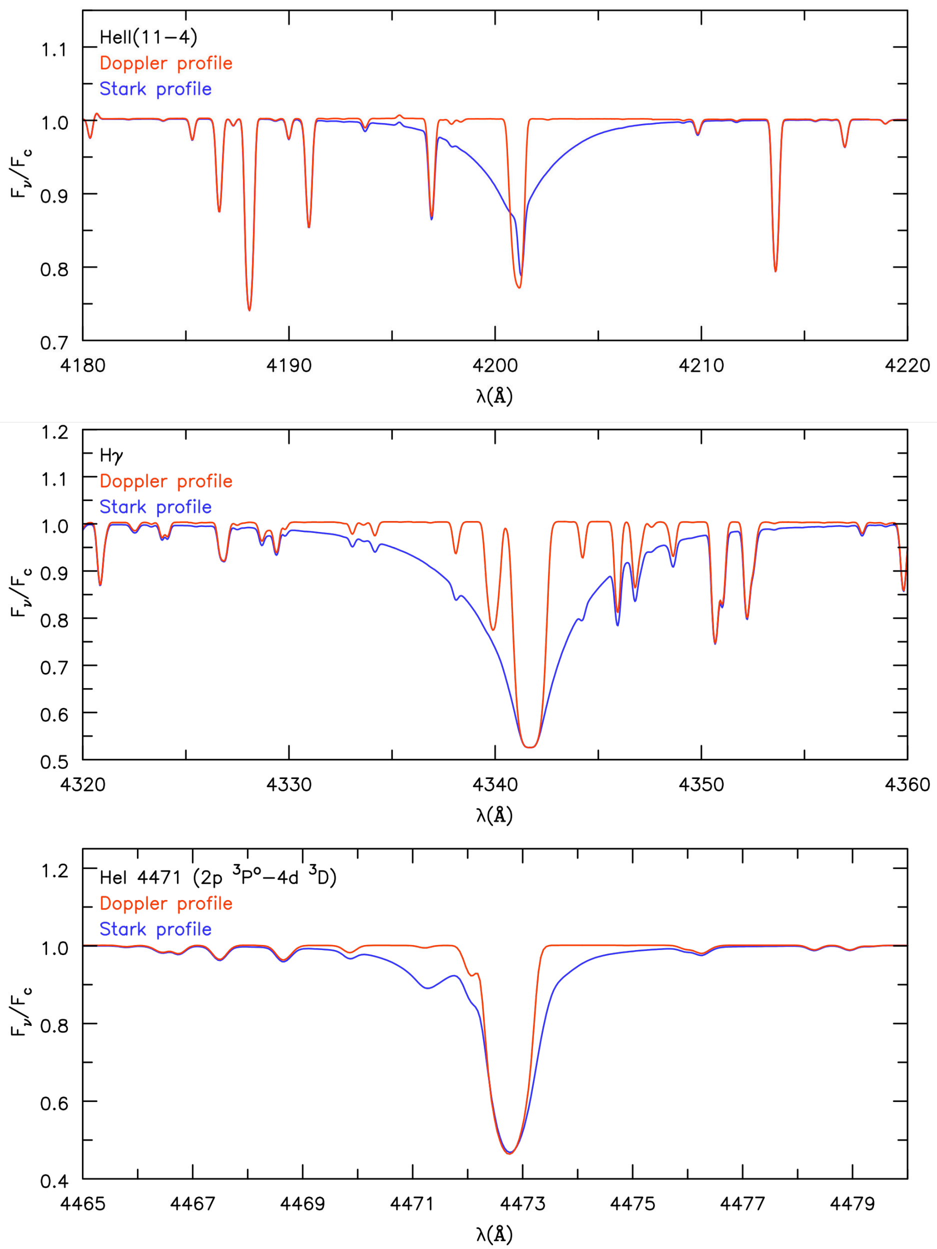
O9 => O Star, effective temperature around 34,000K
V => implies star on main sequence (dwarf).

Luminosity Effects at A0



Increasing surface gravity (log g) and density.

From: http://nedwww.ipac.caltech.edu/level5/Gray/frames.html

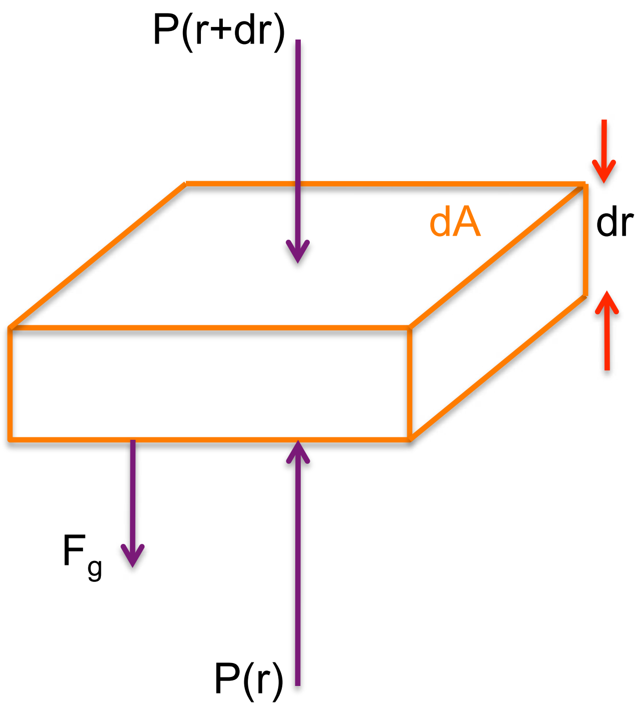


Stark broadening has a major influence on the observed line profiles of H, He I, and He II.

It is particular important for H and He II because the *l* states (i.e., s, p, d etc), for a given n, are degenerate.

Reasons for Plane Parallel Approximation

Real stars are spherical. However, we can approximate their atmospheres as plane-parallel if the thickness of the atmosphere is much less than the radius of the star. To determine the atmosphere thickness we consider the forces acting on a small element inside the star.



The mass of the element of area dA is ρ.dA.dr where ρ is the density. The only forces acting on the element are gas pressure, and gravity. If the star is to be in equilibrium, these forces must be equal.

Define M_r as mass interior to radius *r*.

Thus

$$P(r)dA = P(r+dr)dA + \frac{GM_r}{r^2}.\underbrace{(\rho dAdr)}_{dm}$$

Hence

$$P(r+dr) - P(r) = -\frac{GM_r}{r^2}\rho dr$$

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

This is the EQUATION of HYDROSTATIC EQUILIBIRUM. It provides (with other equations) a means of determining the structure of the star).

We assume only gas pressure was important. This is true for stars like the Sun, but for hot (or very luminous) stars radiation pressure MUST be included in the equation of hydrostatic equilibrium. In the present context we are interested only in the atmospheric layers, and hence $M_r = M_*$. We will also assume the $r = R_*$, the validity of which will be checked. Using z as the dependent variable we have

$\frac{dP}{dz} = -g\rho$ where $g = \frac{GM_*}{R_*^2}$ is the star's SURFACE GRAVITY. The pressure is given by the ideal gas law: $P_g = N_p kT$

where N_p is the particle density,

k is Boltzmann's constant, and

T is the gas temperature.

The density is given by $\rho = \mu_p m_H N_p$ where we use m_H to denote the atomic mass unit, and μ_p is the mean particle mass in AMU (atomic amass units).

Thus

$$\frac{dN_pT}{dz} = -\frac{g\mu_p m_H}{k} N_p$$

To estimate the atmospheric scale height we will assume the atmosphere is isothermal (the density generally varies much more the the temperature). Thus

$$\frac{1}{N_p}\frac{dN_p}{dz} = -\frac{g\mu_p m_H}{kT}$$

which has solution (by integrating from z to infinty)

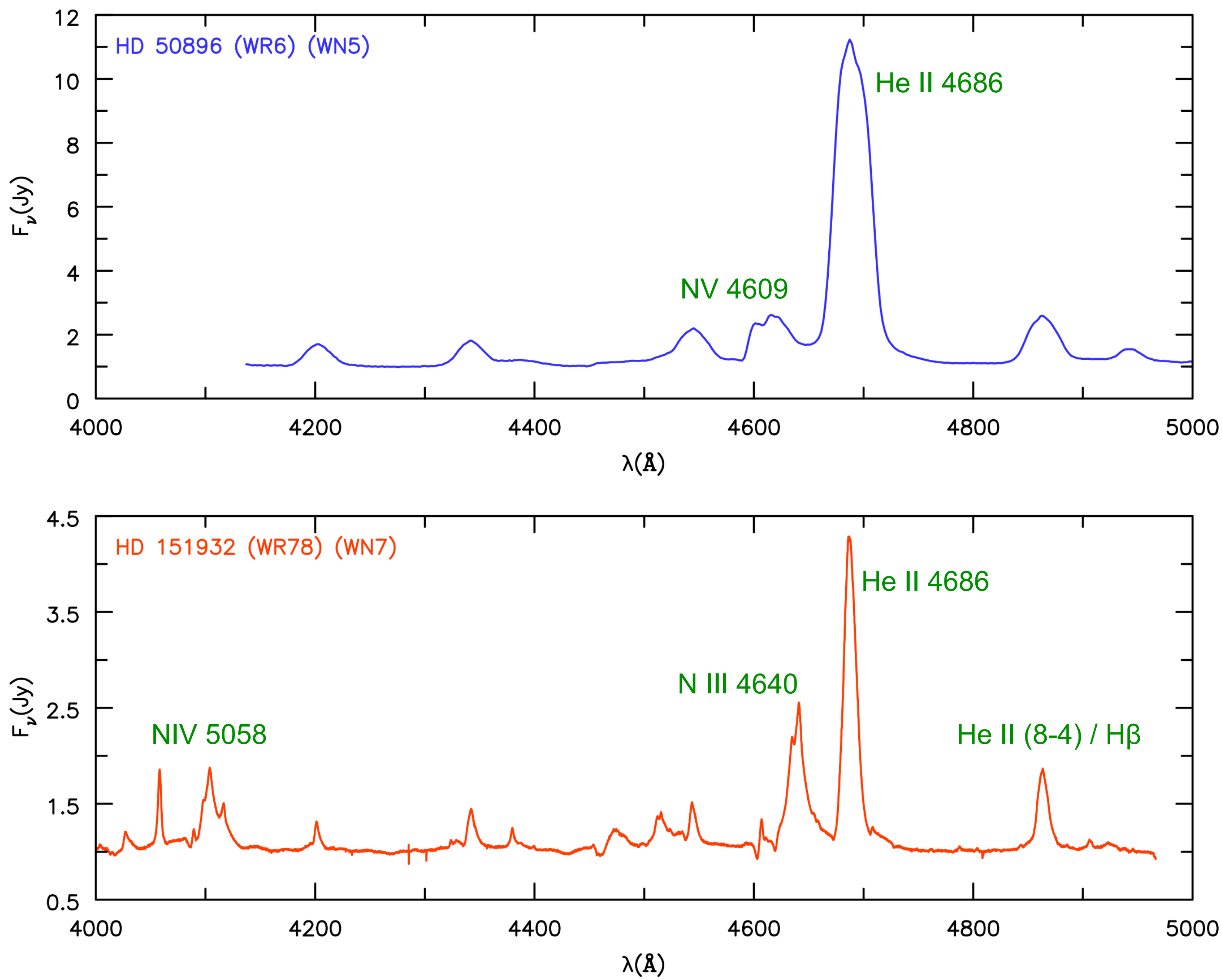
$$N_p = N_o \exp(-z/h)$$

where N_o is the particle density at $z=0$, and h is the atmospheric scale height, and is given by

$$h = \frac{kT}{g\mu_p m_H}$$

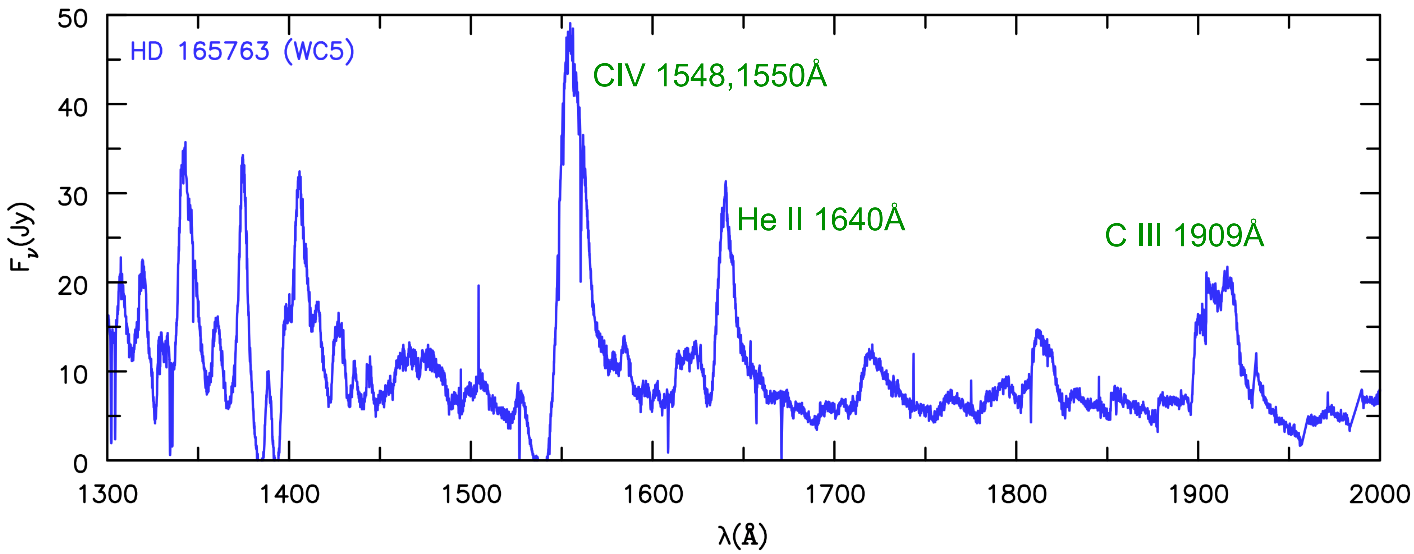
WN Stars

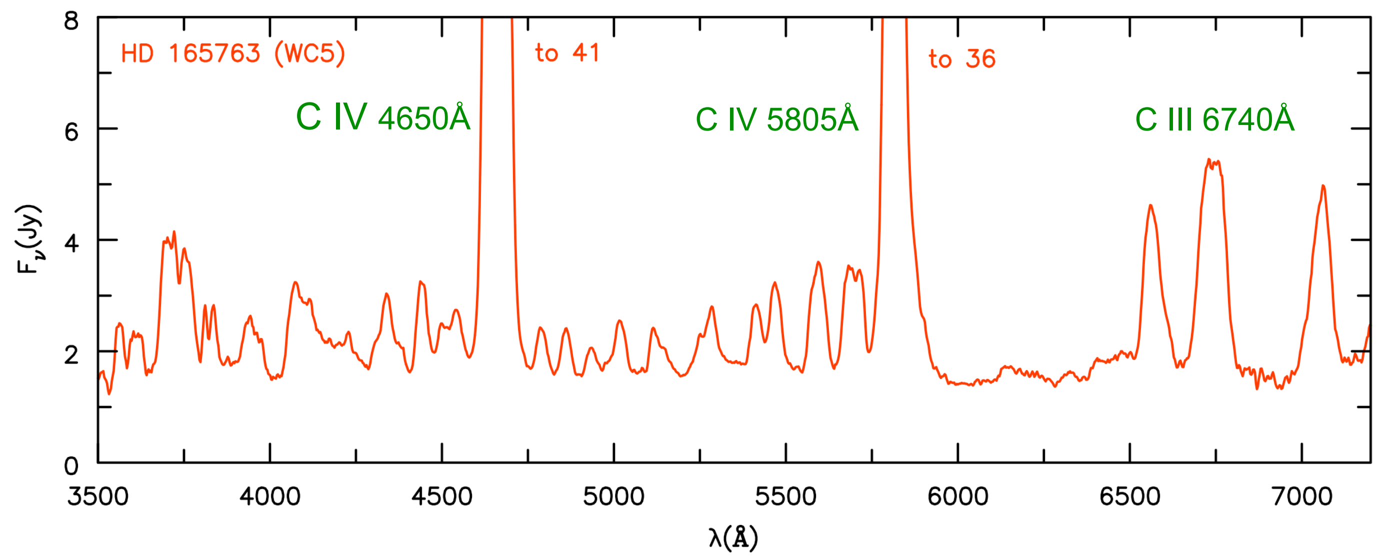
He, N enriched / H, C, O deficient
CNO processed material at surface.



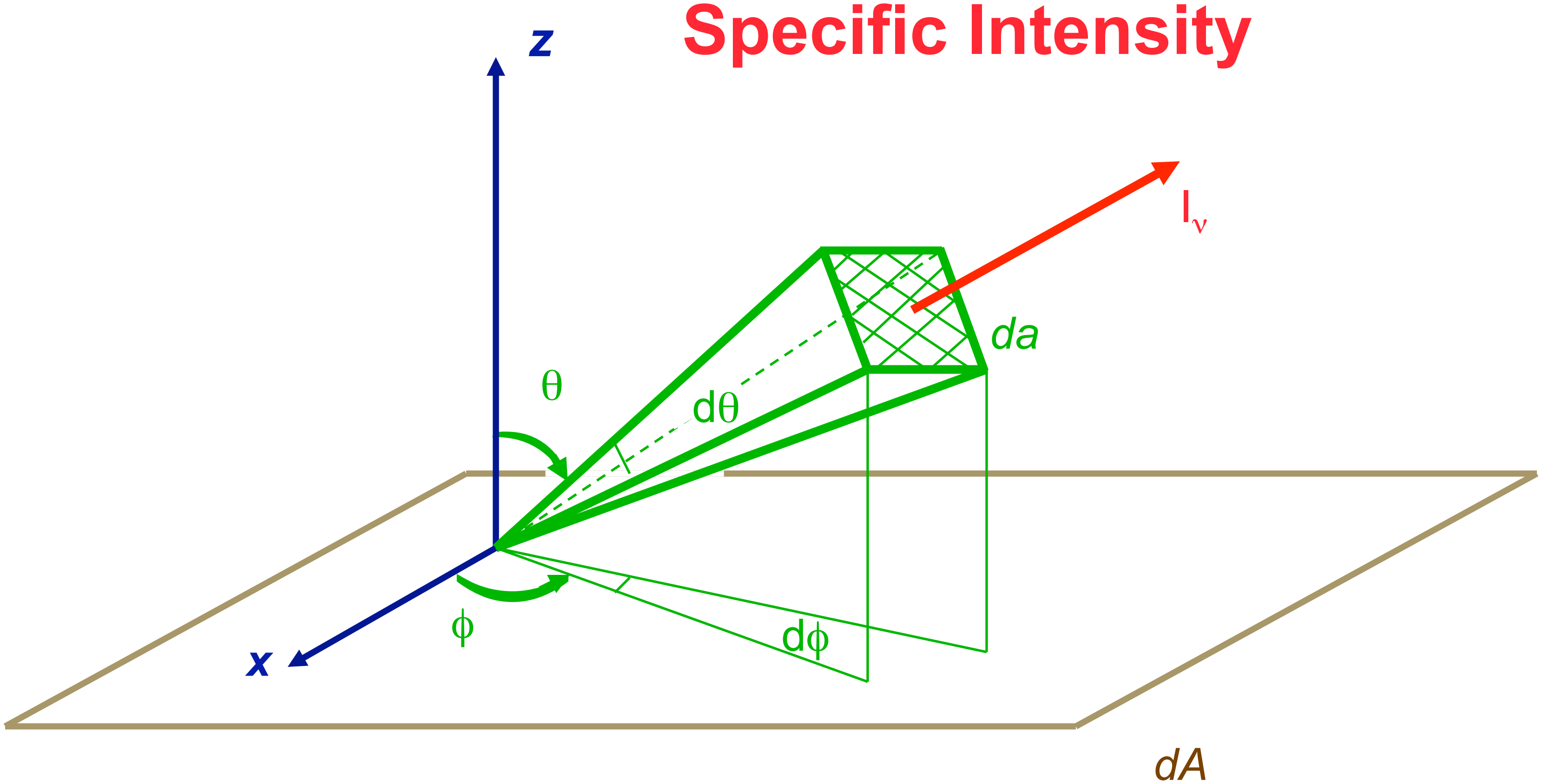
WC Stars

He, C enriched / Zero H, N
He burning material at surface.





Specific Intensity



where

I_ν = specific intenstity (ergs cm^{-2} s^{-1} Hz^{-1} str^{-1})

$d\Omega$ = sin $d\theta$ $d\phi$ = $d\mu$ $d\phi$ (=da/r^2=*solid angle*)

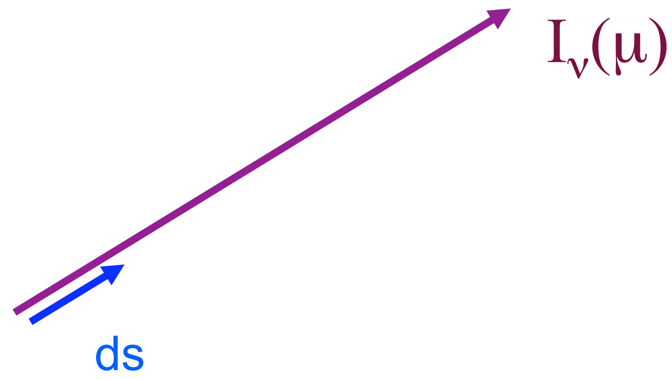
dE = energy passing through dA in direction θ (measured from the surface normal) in time dt, in solid angle $d\Omega$ and in frequency interval $d\nu$.

$$dE = I_\nu dA \cos\theta\, d\nu\, d\Omega\, dt$$

Transfer Equation

Geometry-Free Transfer Equation

$$\frac{dI_\nu}{ds} = \eta_\nu - \chi_\nu I_\nu$$



$$\frac{dI_\nu}{d\tau} = I_\nu - S_\nu$$

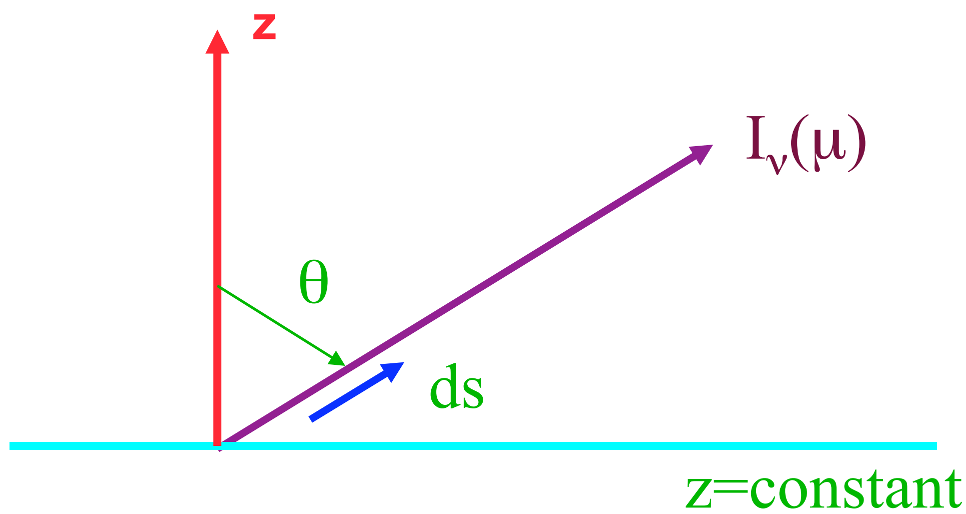
where

- I_ν = specific intensity (ergs cm^{-2} s^{-1} Hz^{-1} sr^{-1})
- s = distance along path
- η_ν = emissivity (ergs cm^{-3} s^{-1} Hz^{-1} sr^{-1})
- χ_ν = opacity (cm^{-1})
- S_ν = source function (ergs cm^{-2} s^{-1} Hz^{-1} sr^{-1})
- τ = optical depth ($d\tau = -\chi ds$)

Plane-Parallel Transfer Equation

$$\mu \frac{dI_\nu}{dz} = \eta_\nu - \chi_\nu I_\nu$$

$$\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu$$



where

z = spatial coordinate (directed outward, and normal to surface)
μ = cos θ
τ = optical depth ALONG z
NB: $d\tau$ (ray) = $-\chi\, dz/\mu$

NB: Radiation field is assumed to be independent of azimuth (ϕ)

Moments of the Radiation Field

Moments of the radiation field

$$J_\nu = \tfrac{1}{2}\int_{-1}^{1} I_\nu(\mu)\,d\mu \qquad H_\nu = \tfrac{1}{2}\int_{-1}^{1} I_\nu(\mu)\,\mu\,d\mu$$

$$K_\nu = \tfrac{1}{2}\int_{-1}^{1} I_\nu(\mu)\,\mu^2 d\mu \qquad N_\nu = \tfrac{1}{2}\int_{-1}^{1} I_\nu(\mu)\,\mu^3 d\mu$$

Physical associations

$$E_\nu = \frac{4\pi}{c} J_\nu \quad \text{(Energy Density)}$$

$$F_\nu = 4\pi\, H_\nu \quad \text{(Energy Flux)}$$

$$P_\nu = \frac{4\pi}{c} K_\nu \quad \text{(Radiation Pressure)}$$

Moment Transfer Equations

Plane Parallel

$$\frac{dH_\nu}{dz} = \chi_\nu (S_\nu - J_\nu)$$

$$\frac{dH_\nu}{d\tau_\nu} = J_\nu - S_\nu$$

$$\frac{dK_\nu}{dz} = -\chi_\nu H_\nu$$

$$\frac{dK_\nu}{d\tau_\nu} = H_\nu$$

Spherical

$$\frac{1}{r^2}\frac{dr^2 H_\nu}{dz} = \chi_\nu (S_\nu - J_\nu)$$

$$\frac{dr^2 H_\nu}{d\tau_\nu} = r^2 (J_\nu - S_\nu)$$

$$\frac{1}{r^2}\frac{dr^2 K_\nu}{dz} + \frac{K_\nu - J_\nu}{r} = -\chi_\nu H_\nu$$

$$\frac{dr^2 K_\nu}{d\tau_\nu} - \frac{r^2 (K_\nu - J_\nu)}{\chi_\nu r} = r^2 H_\nu$$

Notation

Formal Solution.

Solve the transfer equation for I, with J and S known ---- FORMAL SOLUTION.

Timing per frequency:

Plane parallel : $\sim N_\mu . N_D$

Spherical : $\sim N_D^2$ (as # of angle's prop. to N_D)

Use the formal solution to compute f (K/J), g (= N/H).

Moment Solution

Solve for J (given thermal emission and opacity)

Timing per frequency:

$\sim N_D$

Linearized Moment Solution

Solve for δJ

Timing per frequency:

$\sim N_D^2$ (other factors)

Solution of Moment Equations

Choose grid (> 5 points per log τ).

Define J, K on grid nodes [i=1, outer boundary, i=N_D inner boundary]

Define H at mid points.

Thus

$$\frac{H_{i+1/2} - H_{i-1/2}}{\Delta\tau_i} = J_i - S_i$$

$$\frac{f_{i+1}J_{i+1} - f_i J_i}{\Delta\tau_{i+1/2}} = H_{i+1/2}$$

where

$$\begin{aligned} \Delta\tau_{i+1/2} &= 0.5(z_i - z_{i+1})(\chi_{i+1} + \chi_i) \\ \Delta\tau_i &= 0.5(\Delta\tau_{i+1/2} + \Delta\tau_{i-1/2}) \end{aligned}$$

Thus

$$\frac{f_{i+1}J_{i+1}}{\Delta\tau_i \Delta\tau_{1+1/2}} - \left(1 - \theta_i + \frac{f_{i+1}}{\tau_i \Delta\tau_{1+1/2}} + \frac{f_i}{\Delta\tau_i \Delta\tau_{i-1/2}}\right) J_i + \frac{f_{i-1}J_{i-1}}{\Delta\tau_i \Delta\tau_{i-1/2}} = -\zeta_i$$

where

$S_i = \zeta_i + \theta_i J_i$ (allows electron scattering to be taken into account)

These equations can be written in matrix form:

$$\underline{\underline{T}} \cdot \underline{J} = \underline{S}$$

where $\underline{\underline{T}}$ is a $N_D.N_D$ tridiagonal matrix, and $\underline{J}$ and $\underline{S}$ are column vectors (length N_D). Solved using Thomas Algorithm.

Boundary conditions: next slide

Linearization.

$$\frac{f_{i+1}\delta J_{i+1}}{\Delta\tau_i \Delta\tau_{1+1/2}} - \left(1 - \theta_i + \frac{f_{i+1}}{\tau_i \Delta\tau_{1+1/2}} + \frac{f_i}{\Delta\tau_i \Delta\tau_{i-1/2}}\right)\delta J_i + \frac{f_{i-1}\delta J_{i-1}}{\Delta\tau_i \Delta\tau_{i-1/2}} = F_{i-1}\delta\chi_{i-1} + F_i \delta\chi_i + F_{i+1}\delta\chi_{i+1} + G_i \delta\eta_i + R_i \delta\theta_i$$

In matrix form

$$\underline{\underline{T}} \cdot \underline{dJ} = \underline{\underline{F}} \cdot \underline{d\chi} + \underline{\underline{G}} \cdot \underline{d\eta} + \underline{\underline{R}} \cdot \underline{d\theta}$$

where $\underline{F}$, $\underline{G}$ and $\underline{R}$, are $N_D.N_D$ matrices (not full), and $\underline{dJ}$, $\underline{d\chi}$, $\underline{d\eta}$, $\underline{d\theta}$ are column vectors of length N_D. This is easily solved for $\underline{dJ}$ (in terms of $\underline{d\chi}$, $\underline{d\eta}$, $\underline{d\theta}$).

Notes

Accuracy

Equations are second order accurate.

Inner boundary

Use diffusion approximation.

Outer boundary.

For plane-parallel, and spherical atmosphere can formulate a second order boundary condition for the Moment equations. For the CMF (comoving-frame) transfer equation not feasible. This can cause a loss of accuracy and instabilities. The later is removed by using a VERY SMALL step size at outer boundary. The instability is complicated and can cause J to blow up.

Choosing I(inward)=0 at outer boundary can cause problems as very difficult to extend atmosphere so that $\tau = 0$ at all wavelengths. Can choose a very fine grid at outer boundary, but unphysical.

In CMGEN we extrapolate the opacities/emissivities at the outer boundary. This gives an I(inward) we can be used to define a b.c. for the outer boundary.

Diffusion Approximation

$$I_\nu = B_\nu + \mu \frac{dB_\nu}{d\tau_\nu}$$

$$J_\nu = 3K_\nu = B_\nu$$

$$H_\nu = \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} = \frac{1}{3\chi_\nu}\frac{dB_\nu}{dT}\left|\frac{dT}{dz}\right|$$

$$\left|\frac{dT}{dR}\right| = \frac{3L}{64\pi r^2 \sigma T^3}\bar{\chi}_{Rosseland} = \frac{3}{16}\left(\frac{T_{eff}}{T}\right)^3\left(\frac{R_*}{r}\right)^2 T_{eff}\bar{\chi}_{Rosseland}$$

$$\bar{\chi}_{Rosseland} = \frac{\int \frac{dB_\nu}{dT} d\nu}{\int \frac{1}{\chi}\frac{dB_\nu}{dT} d\nu}$$

When diffusion approximation is valid:

Flux is transmitted where opacity is lowest.

At each depth, radiation transport is described by a single number --- the Rosseland mean opacity.

Saha- Boltzmann Equation
(applies when Local Thermodynamic Equilibrium [LTE] is valid)

Relation between two level populations in the same ionization stage.

$$\frac{N_u^*}{N_l^*} = \frac{g_u}{g_l}\exp\left(-h\nu_{lu}/kT\right)$$

N_l= population density (cm^{-3}) of lower energy level, l

g_l= degeneracy of state l
= statistical weight

$h\nu_{lu}$= difference in energy

(* denotes LTE)

Relation between two populations in different (but consecutive) ionization stages.

$$N_{\kappa,j}^* = N_e^* N_{\kappa+1,1}^* \frac{g_{\kappa,j}}{g_{\kappa+1,1}} \frac{C_I}{T^{1.5}} \exp(\psi_{\kappa,j}/kT)$$

$N_{\kappa,j}$ = population of level j in ionization state κ
$N_{\kappa+1,1}$ = population of level 1(g.s.) in ionization state κ+1
$g_{\kappa,j}$ = degeneracy of level j in ionization state κ
$g_{\kappa+1,j}$ = degeneracy of g.s. in ionization state κ+1
$\psi_{\kappa,j}$ = ionization energy of state j

What is the LTE level population when LTE is not-valid?

Relation between two level populations in the same ionization stage.

$$\frac{N_u^*}{N_l^*} = \frac{g_u}{g_l}\exp\left(-h\nu_{lu}/kT\right)$$

N_l= population of lower energy level, *l*
g_l= degeneracy of state *l*
= statistical weight
$h\nu_{lu}$= difference in energy

The LTE ratio of two levels in the same ion is well defined.

What is the LTE level population when LTE is not-valid?

Define the LTE population by the Saha equation using the actual ion density and the actual electron density.

$$N^*_{\kappa,j} = N_e N_{\kappa+1,1} \frac{g_{\kappa,j}}{g_{\kappa+1,1}} \frac{C_I}{T^{1.5}} \exp(\psi_{\kappa,j}/kT)$$

However, there is no longer a unique LTE definition. Can define LTE, for example, with respect to another level in the next ionization stage.

$$\overline{N}^*_{\kappa,j} = N_e N_{\kappa+1,u} \frac{g_{\kappa,j}}{g_{\kappa+1,u}} \frac{C_I}{T^{1.5}} \exp([\psi_{\kappa,j} + h\nu_{\kappa+1,1u}]/kT)$$

If we define

$$b_{\kappa+1,u} = N_{\kappa+1,u} / N^*_{\kappa+1,j}$$

then

$$\overline{N}^*_{\kappa,j} = \frac{b_{\kappa+1,u}}{b_{\kappa+1,1}} N^*_{\kappa,j}$$

Sources of Opacity

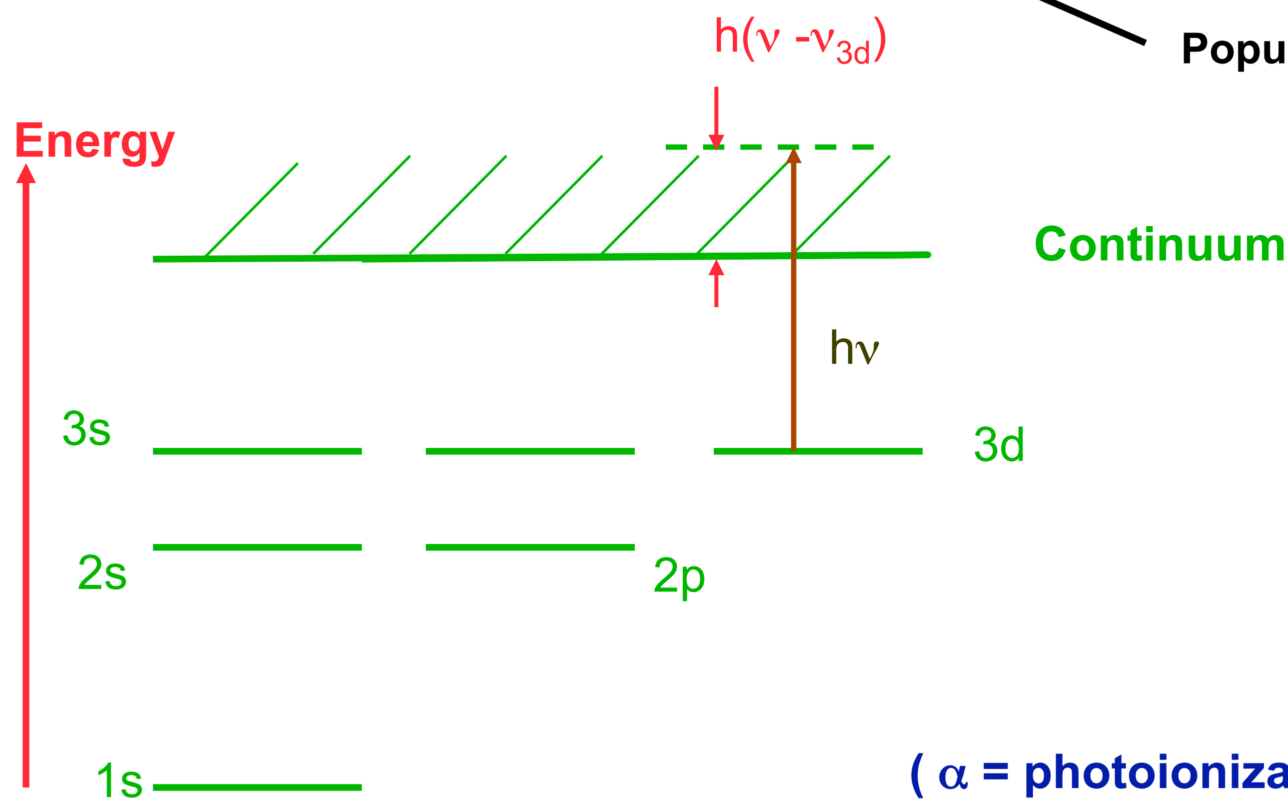
Bound-free

$$\chi_j(\nu) = \alpha(\nu)_j \left(N_j - N_j^* \exp(-h\nu / kT) \right)$$

Cross-section (cm²) – set by atomic physics.

LTE population of level j

Population of level j



(α = photoionization cross-section)

Ionizations to excited states

Consider C III (i.e., C2+). Then

C III 2s nl + hv <--> C IV 2s + e^-

but we also can have

C III 2p nl + hv <--> C IV 2p + e^-

The expression

$$\chi_j(\nu) = \alpha_\nu \left(N_j - N_j^* \exp(-h\nu / kT) \right)$$

is valid for both processes, provided we use the the correct LTE population --- the LTE that is defined with respect to the actual ion involved (i.e, C IV 2s of C IV 2p).

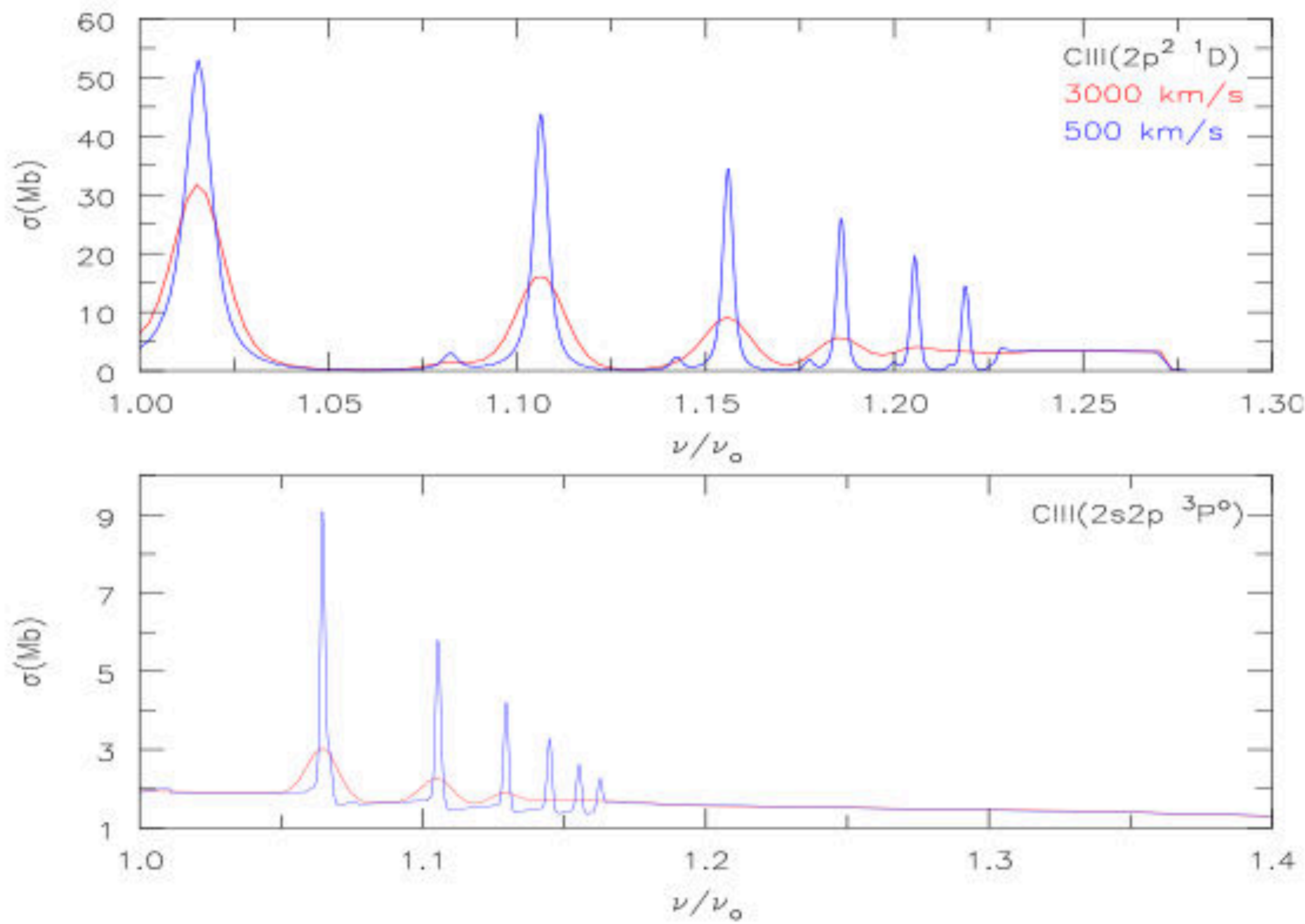
Resonance in Photoionization Cross-sections

Multi-electron systems can have states that lie above the ionization limit. If the state satisfies certain selection rules (e.g., parity) this state can autoionize to the ion and a free electron. The autoionization rates can be large (10^{14}) and thus the doubly exited state is in LTE with respect to the ion. Instead of autoionizing, the state can undergo a bound-bound decay which moves it below the ionization limit, and hence cause a ``recombination''. The reverse process can also occur, and appear as resonances in the photoionization cross-section.

e.g. CIII 2p nl : For high b, 2p electron decays. For these states to be populated, T large (although not very large for CIII) and the process is referred to as HTDR (high temperature dielectronic recombination).

When the 2p nl state is very close to the ionization limit (which occurs for low n e.g., 4) the state can provide an effective recombination rate at low temperatures. Rates are sensitive to the precise location of the atomic energy levels, and hence atomic physics. For many of the levels, the decay of the nl state may provide the dominate recombination route (Low TDR)

Photoionization Cross-section



LTDR can be treated assuming the autoionizing levels are bound states, or by treating them as a resonance in the photoionization cross-section. The first peak in the top figure correspond to the dielectronic recombination line at 412Å.To avoid aliasing, the photoionization cross-sections are smoothed.